Mathematics 528
Ergodic Theory/Topological Dynamics
Spring Semester, 2016

SYLLABUS

Course Objectives. A lecture course devoted to understanding the interplay of number theory and ergodic theory/topological dynamics in an important paper by Peter Sarnak.

Grading: Grades will be based upon attendance and class participation.

Absences: As indicated above, attendance will be an important part of the grades in this course.

Textbook: There will be no assigned text in this course. Notes covering the topics to be covered (and then some) will be distributed. The introduction to these notes, which is included in this syllabus, gives a more complete description of the goals of the course.

Examinations and papers: No examinations or required papers are expected in this course.

Assignments: No required assignments are expected in this course.

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Office Hours: By drop-in or appointment.

Any student with a disability that requires accomodation is encouraged to contact the instructor and Disability Support Sevices in the Allen Center.

The information contained in this syllabus, other than the absence policy, may be subject to change, with reasonable advance notice, as deemed appropriate by the instructor.

Mathematics 528*

Möbius Dynamics

Spring Semester, 2016

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1 Introduction.

In his seminal paper, [Sa11], Sarnak establishes a striking connection between notions in number theory, relating to deep open questions about the primes, and notions in well-developed areas of dynamical systems and ergodic theory. The purpose of these notes is to describe some of the issues involved.

Let Π denote the set of prime numbers, and recall that the **Möbius function** is the sequence, $\mu(n), n \in \mathbb{N}$, defined by

$$\mu(n) = \begin{cases} 0, p^2 | n, \text{ some } p \in \Pi \\ 1, n = 1 \\ (-1)^r, r > 0, n = p_1 \cdot p_2 \cdots p_r, p_i \in \Pi, p_i \neq p_j, i \neq j \end{cases} \tag{1.1}$$

Among other things, $\mu(\cdot)$ is a sequence with values in $\{0, \pm 1\}$ and therefore a point of the compact space

$$\Omega_3 = \{0, \pm 1\}^{\mathbb{N}}.$$

Equip this space with the **left shift**,

$$(\sigma\omega)(n) = \omega(n+1), n > 0, \omega \in \Omega_3.$$

Denote by $\mathcal{A}_3$ the closure of the images of the point (1.1) under powers $\sigma^k, k \geq 0$. Chowla, [Ch65], conjectured that if $s > 0$, then for all choices $0 \leq k_1 < k_2 < \cdots < k_s$ and $a_1, a_2, \cdots, a_s \in \{1, 2\}$ with at least one $a_i = 1$, it will be true that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left(\prod_{j=1}^{s} \mu(n + k_j)^{a_j} \right) = 0. \tag{1.2}$$

Sarnak's first observation is that (1.2) is tantamount to the existence of a Borel probability measure, M, on $\mathcal{A}_3$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) f(\sigma^n \mu) = \int_{\mathcal{A}_3} f(\omega) M(d\omega), f \in C(\mathcal{A}_3), \tag{1.3}$$

the limit existing. One says that μ is a **generic point** for the measure M. (Section 21 ff., of these notes.)

Define a set $\mathcal{A} \subset \{0,1\}^{\mathbb{N}}$ as the image of $\mathcal{A}_3$ under the "squaring map", τ, where

$$(\tau\omega)(n) = \omega(n)^2, n > 0, \omega \in \Omega_3. \tag{1.4}$$

Remark 1.1 *While by definition $\mathcal{A}_3 \subseteq \tau^{-1}\mathcal{A}$, it is a deep question whether equality holds. For example, the Chowla Conjecture implies equality holds. Sarnak gives an elegant number-theoretic description of $\mathcal{A}$, which is described in Sections 5 and 6 of these notes. It has been suggested ([H86]) that the case $s = 2$, $k_1 = 0, k_2 = 1$ and $a_1 = a_2 = 1$ of the Chowla Conjecture (1.2) may be on a level similar to that of the twin prime conjecture.*

Set $\varkappa = \tau\mu$, and observe that on the Chowla conjecture, $\varkappa$ is a generic point for the image measure

$$\Lambda_\varkappa = \tau M. \tag{1.5}$$

However, as Sarnak observes, one does not require the Chowla conjecture in order to prove that $\varkappa$ is generic for **some** measure, which we may as well denote by $\Lambda_\varkappa$. This fact is informed by classical works, [Mi47], [Hall82], [Hall89] and [Ts85]. A nontrivial fact is that the closed support of $\Lambda_\varkappa$ is **all** of $\mathcal{A}$. (Section 11 of these notes.)

Sarnak also observes that the flow $(\mathcal{A}, \mathcal{B}, \Lambda_\varkappa, \sigma)$ is **ergodic**, meaning that the only measurable functions a.e. invariant under σ are the constant a.e. functions. (Ergodicity is proved in Section 15 of these notes.) Sarnak constructs a map, ρ,

$$(\mathcal{A}, \mathcal{B}, \Lambda_\varkappa, \sigma) \xrightarrow{\rho} (K, \mathcal{B}, \lambda, T). \tag{1.6}$$

To explain, the image of the map is contained in the compact, abelian topological group, actually ring,

$$K = \prod_{p \in \Pi} \mathbb{Z}_{p^2}, \mathbb{Z}_{p^2} = \mathbb{Z}/\left(p^2\mathbb{Z}\right). \tag{1.7}$$

Algebraic operations are coordinatewise. If θ is the point all of whose coordinates are 1, i.e., the multiplicative unit in the ring, then

$$T\gamma = \gamma - \theta, \gamma \in K.$$

λ is the unique T-invariant Borel probability measure on K, and Sarnak observes that his map satisfies

$$\rho \circ \sigma = T \circ \rho$$

$$\rho \Lambda_{\varkappa} = \lambda.$$

(Sarnak's map is described in Section 17 of these notes.)

Cellarosi-Sinai, [CS13], prove that Sarnak's map is a.e. injective, a measure-theoretic isomorphism between the processes (1.6). (The Cellarosi-Sinai Theorem is proved in Section 15 of these notes.) Informed by the Cellarosi-Sinai work and the works by Mirsky, et al. cited above, it is possible to express the function $\varkappa(\cdot)$ as a "Fourier series",

$$\varkappa(n) = \tfrac{6}{\pi^2} \sum_{m=1}^{\infty} \mu(m) A(m) P_m(n)$$

$$A(m) = \left(\prod_{p|m} (p^2 - 1) \right)^{-1} \tag{1.8}$$

$$P_m(n) = \sum_{\vec{i} \in \mathbb{Z}^*_{m^2}} e^{2\pi i \theta_m(\vec{i})n}.$$

For square-free $m, i.e., \mu(m) \neq 0$, the Chinese Remainder Theorem (Section 3 of these notes) implies $\mathbb{Z}_{m^2}$ may be identified with the product group

$$\mathbb{Z}_{m^2} = \prod_{p|m} \mathbb{Z}_{p^2}.$$

$\mathbb{Z}^*_{m^2} \subset \mathbb{Z}_{m^2}$ and $\theta_m\left(\vec{i}\right)$ are defined by

$$\mathbb{Z}^*_{m^2} = \left\{ \vec{i} \in \prod_{p|m} \mathbb{Z}_{p^2} | 1 \leq i_p < p^2, p|m \right\} \tag{1.9}$$

and

$$\theta_m\left(\vec{i}\right) = \sum_{p|m} \tfrac{i_p}{p^2}, \ \vec{i} \in \mathbb{Z}^*_{m^2}.$$

The sense of convergence in (1.8) is in the sense of Besicovitch almost periodic functions (Section 7 of these notes). The representation (1.8) is established in Section 14 of these notes. The observation that $\varkappa(\cdot)$ is Besicovitch almost periodic was made by Rauzy, [R65].

Sarnak's Conjecture bears some similarity to the Chowla Conjecture. He introduces a certain notion of determinism of bounded functions on the natural numbers and suggests it should be true for any one of these that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(n)f(n)=0. \tag{1.10}$$

The conjecture is stated more precisely in Section 23, but a brief précis will be given below. To illustrate the apparent depth of Sarnak's conjecture, the periodic functions belong to the as yet undefined deterministic class. For periodic functions, (1.10) has long been known to be equivalent to the prime number theorem for arithmetic progressions. Nevertheless, as Sarnak states, his conjecture is apparently weaker than the Chowla Conjecture. Section 23 is devoted to proving Sarnak's Theorem, that his conjecture would be a consequence of Chowla's conjecture. While many instances of Sarnak's Conjecture have been established, by Sarnak, his collaborators and others, the complete conjecture seems far from being established. In this regard, Section 26 discusses one open case that promises to be difficult to resolve. A tool that has proved to be instrumental in the resolution of many cases is a powerful lemma by Bourgain-Sarnak-Ziegler, [BSZ2011}, to wit

Theorem 1.2 *Katai-Bourgain-Sarnak-Ziegler ([K86],[BSZ2011]) Let* $h\in l^{\infty}(\mathbb{N})$. *If*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}h(pn)h(qn)=0$$

for all but a finite number of distinct primes, p,q, *then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(n)h(n)=0.$$

The class of deterministic functions to which the Sarnak conjecture pertains is a class of "orbit functions". Let Y be a compact metric space

equipped with a continuous self map and a point $y_0 \in Y$. Every $\varphi \in C(Y)$ determines a bounded function on $\mathbb{N}$, $f(n) = \varphi(T^{n-1}y_0)$. One calls the point y_0 **completely deterministic** if all weak-$*$ cluster points, υ, of the sequence of atomic probability measures

$$\upsilon_N = \frac{1}{N}\sum_{n=1}^{N}\delta_{T^{n-1}y_0}, \tag{1.11}$$

which automatically govern measure preserving transformations, (Y, T, υ), have measure entropy, $h_\upsilon(T) = 0$. Any cluster point of the sequence (1.11) is necessarily T-invariant. Measure entropy is discussed in the Appendix to these notes, Appendices A-H.

For example, the flows $(\mathcal{A}, \sigma, \Lambda_\varkappa)$ and $(\mathcal{A}_3, \sigma, M)$ are measure preserving. The former has $h_{\Lambda_\varkappa}(\sigma) = 0$ (Section 15), while the latter satisfies $h_M(\sigma) = \frac{6}{\pi^2}\log 2$ (Section 20). Recall the natural map

$$\begin{array}{c} (\mathcal{A}_3, \sigma, M) \\ \downarrow \tau \\ (\mathcal{A}, \sigma, \Lambda_\varkappa) \end{array} \tag{1.12}$$

Not only is $h_{\Lambda_\varkappa}(\sigma) = 0$, but (1.12) exhibits the **largest** zero entropy factor of $(\mathcal{A}_3, \sigma, M)$. This means that given a homomorphism

$$\begin{array}{c} (\mathcal{A}_3, \sigma, M) \\ \downarrow \chi \\ (W, R, \eta) \end{array} \tag{1.13}$$

with $h_\eta(R) = 0$, the pullback σ-algebra $\chi^{-1}\mathcal{B}(W) \subseteq \mathcal{B}(\mathcal{A}_3)$ **must be** a subalgebra of the pullback "Pinsker algebra" $\tau^{-1}\mathcal{B}(\mathcal{A})$. (Sections 22-23 and Appendix H). This means there is a commutative triangle of homomorphisms

$$\begin{array}{ccc} (\mathcal{A}_3, \sigma, M) & & \\ \downarrow \tau & \overset{\chi}{\searrow} & , \rho \circ \tau = \chi \\ (\mathcal{A}, \sigma, \Lambda_\varkappa) & \xrightarrow{\rho} & (W, R, \eta) \end{array} \tag{1.14}$$

The putative "Chowla measure", M, may be characterized and proved to exist without the Chowla Conjecture necessarily being true. This is explained in Section 20[1]. The measure, which is supported on $\tau^{-1}\mathcal{A} \supseteq \mathcal{A}_3$, does satisfy (1.12) and (1.13). For convenience of notation in this introduction, we shall write $\mathcal{A}_3$ in place of $\tau^{-1}\mathcal{A}$. (In fact, $\mathcal{A}_3 = \tau^{-1}\mathcal{A}$ is the **definition** of $\mathcal{A}_3$ in Section 20.) Given (Y,T), a completely deterministic point y_0 (which may or may not exist for a given (Y,T)) and $\varphi \in C(Y)$, set up the product space $\mathcal{A}_3 \times Y$ and on it the continuous map $\sigma \times T$. Let $\pi_1(\cdot) \in C(\mathcal{A}_3)$ be the first-coordinate function, and consider the product

$$F(\omega, y) = \pi_1(\omega) \cdot \varphi(y). \tag{1.15}$$

Observe that taking $(\omega, y) = (\mu, y_0)$, one has

$$\begin{aligned} \tfrac{1}{N}\sum_{n=1}^{N} F\left((\sigma \times T)^{n-1}(\mu, y_0)\right) &= \tfrac{1}{N}\sum_{n=1}^{N} \pi_1(\sigma^{n-1}\mu)\,\varphi(T^{n-1}y_0) \\ &= \tfrac{1}{N}\sum_{n=1}^{N} \mu(n)\, f(n),\ f(n) = \varphi(T^{n-1}y_0). \end{aligned} \tag{1.16}$$

One argues, à la Furstenberg, [Furstenberg67], [Furstenberg81], that failure of (1.16) to converge to zero is tantamount to the existence on $\mathcal{A}_3 \times Y$ of a $(\sigma \times T)$-invariant probability measure ξ such that

$$\int_{\mathcal{A}_3 \times Y} F(\omega, y)\, \xi(d(\omega, y)) \neq 0. \tag{1.17}$$

The measure ξ is a cluster point of the sequence indicated by (1.16), and therefore its projection on the second coordinate is a cluster point, denoted υ, of (1.11). The assumption on y_0 implies that $h_\upsilon(T) = 0$. Now if we assume the Chowla Conjecture, the projection of ξ on the first coordinate is M. One speaks of ξ as a **joining** of M and υ. However, there are neither (a) a reason to expect that (Y,T,υ) will be a (measure-theoretic) factor of $(\mathcal{A}_3, \sigma, M)$ nor (b) a consequence to be derived if it is. The argument is a little more subtle at this point.

[1]Section 23, Remark 23.4, addresses a certain issue of invertibility, which is not discussed in this introduction.

In what follows we also use T to denote the natural map that the action (Y,T) induces on the space, $\mathcal{P}(Y)$, of Borel probability measures on Y, which with the weak-$*$ topology is compact and metrizable. See [Furstenberg81], Chapter 5, for properties of "disintegration" invoked below and in Section 19 of these notes.

The measure ξ admits a disintegration over M and subordinate to the projection-on-the-first-coordinate homomorphism

$$(\mathcal{A}_3 \times Y, \sigma \times T, \xi) \to (\mathcal{A}_3, \sigma, M).$$

The disintegration entails an a.e. unique Borel map, $\mathcal{A}_3 \to \mathcal{P}(Y)$ and denoted $\omega \to \upsilon_\omega$, with certain additional properties:

$$T\upsilon_\omega = \upsilon_{\sigma\omega}, \omega \in \mathcal{A}_3$$

$$\int\limits_{\mathcal{A}_3 \times Y} H(\omega, y)\, \xi(d(\omega, y)) = \int\limits_{\mathcal{A}_3} \left(\int\limits_Y H(\omega, y)\, \upsilon_\omega(dy) \right) M(d\omega), H \in C(\mathcal{A}_3 \times Y). \tag{1.18}$$

The inner integrand is (bounded and) Borel measurable on $\mathcal{A}_3$ because $\omega \to \upsilon_\omega$ is Borel. The image of the invariant measure M under the map is a Borel probability measure on $\mathcal{P}(Y)$,

$$\lambda = \upsilon_\omega M(d\omega)$$

$$\lambda(B) = \int\limits_{\mathcal{A}_3} \chi_B(\upsilon_\omega)\, M(d\omega), B \in \mathcal{B}(\mathcal{P}(Y)).$$

Notice that since M is σ-invariant, λ is T-invariant:

$$\lambda(T^{-1}B) = \int\limits_{\mathcal{A}_3} \chi_{T^{-1}B}(\upsilon_\omega)\, M(d\omega)$$

$$= \int\limits_{\mathcal{A}_3} \chi_B(T\upsilon_\omega)\, M(d\omega) = \int\limits_{\mathcal{A}_3} \chi_B(\upsilon_{\sigma\omega})\, M(d\omega) \tag{1.19}$$

$$= \int\limits_{\mathcal{A}_3} \chi_B(\upsilon_\omega)\, M(d\omega) = \lambda(B).$$

If $H(y)$ depends only upon the second coordinate of $\mathcal{A}_3 \times Y$, the fact that ξ projects to υ implies

$$\int_Y H(y)\,\upsilon(dy) = \int_{\mathcal{A}_3} \left(\int_Y H(y)\,\upsilon_\omega(dy) \right) M(d\omega), H \in C(Y). \tag{1.20}$$

One speaks of υ as the **barycenter** of the measure $\lambda = \upsilon_\omega M(d\omega)$, symbolically

$$\upsilon = \int_{\mathcal{P}(Y)} \xi\lambda(d\xi).$$

An invariant probability measure on $\mathcal{P}(Y)$ with barycenter υ is called a **quasifactor** of (Y, T, υ). The critical fact about quasifactors is that a quasifactor of a zero entropy system is a zero entropy system. (See Appendix H of these notes, and, in much more detail, [Glasner2003], especially Chapters 8 and 18 .)

The measure M admits its own disintegration over the homomorphism (1.12). Express this symbolically as

$$M = \int_{\mathcal{A}} M_x \Lambda_\varkappa(dx), \sigma M_x = M_{\sigma x}. \tag{1.21}$$

As indicated above, the Chowla measure M does not require the Chowla Conjecture for its construction. One of its properties (Sections 20-22) is that the first coordinate function, π_1, on $\mathcal{A}_3$ satisfies

$$\int_{\tau^{-1}x} \pi_1(\omega)\,M_x(d\omega) = 0, \Lambda_\varkappa - a.e.\ x \in \mathcal{A}. \tag{1.22}$$

Using the fact that the quasifactor λ of the zero entropy system (Y, T, υ) also has zero entropy, there exists a commutative triangle as in (1.14). Naturality implies that the function $\omega \to \upsilon_\omega$ is $M_x - a.e.\ \omega$ constant on $\tau^{-1}x, \Lambda_\varkappa - a.e.\ x \in \mathcal{A}$. It therefore makes sense to let υ_x represent the a.e. constant value whenever $\omega \to \upsilon_\omega$ is $M_x - a.e.\ \omega$ constant on $\tau^{-1}x$. The disintegration of ξ is now represented **over** $\mathcal{A}$.

$$\xi = \int_{\mathcal{A}} \left(\int_{\tau^{-1}x} (\delta_\omega \times \upsilon_x)\,M_x(d\omega) \right) \Lambda_\varkappa(dx). \tag{1.23}$$

Should $H(\omega, y) = H(\omega) G(y)$ be a product of continuous functions of ω and y, then

$$\int_{\mathcal{A}_3 \times Y} H(\omega, y) \xi(d(\omega, y)) = \int_{\mathcal{A}} \left(\int_{\tau^{-1}x \times Y} H(\omega) G(y) (M_x \times \upsilon_x)(d(\omega, y)) \right) \Lambda_\varkappa(dx)$$

$$= \int_{\mathcal{A}} \left(\int_{\tau^{-1}x} H(\omega) M_x(d\omega) \right) \left(\left(\int_Y G(y) \upsilon_x(dy) \right) \right) \Lambda_\varkappa(dx). \tag{1.24}$$

In the particular case of $F(\omega, y) = \pi_1(\omega) \cdot \varphi(y)$ in (1.15), (1.22) implies the first factor has integral zero for a.e. x. Therefore, (1.17) cannot be true. In broad outline this is how the Chowla Conjecture implies the Sarnak conjecture.

Acknowledgement 1.3 *The reader is encouraged to become acquainted with the important books, [Glasner2003], [Furstenberg81], [Pa69] and [W82]. In addition, there are many papers that have been critical for these notes, some of which have spoken to generalizations of the Sarnak Conjecture. Most important, of course, is [Sa11]. Other important papers have been [CS13], [P13], [BSZ2011], [Furstenberg67], [LS13] and [AKLR14]. A central issue for one case in [Sa11], Section 24 of these notes, is the classical theorem of Davenport about exponential sums, [D-37II], which is covered in [IK04], Theorem 13.10.*

2 Disjointness of Groups.

Let there be given groups, G_1, G_2. Let $G = G_1 \times G_2$ be the product group. Use $\pi_j, j = 1, 2$, to denote the projection on the j^{th} coordinate. Denote by $e_j, j = 1, 2$, the respective identity elements. $e = (e_1, e_2)$ is the identity element of G.

Let $H \subseteq G$ be a subgroup that is **full**, in the sense that $\pi_j H = G_j, j =$

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